Period 4, Jan 16, 2025

Use properties of natural logarithms and fundamental trigonometric identities to show that the pair of expressions is equivalent.

$\ln|\tan x| + \ln|\cos x|$ and $\ln|\sin x|$

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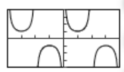
on Board

In the following exercise, half of an identity and the graph of this half are given. Use the graph to make a conjecture as to what the right side of the identity should be. Then prove your conjecture.

$$\frac{(\csc x+\cot x)(\csc x-\cot x)}{\sin x} = ?$$

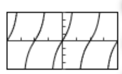


$[-2\pi, 2\pi, \pi/2]$ by $[-4, 4, 1]$

on Board

In the following exercise, half of an identity and the graph of this half are given. Use the graph to make a conjecture as to what the right side of the identity should be. Then prove your conjecture.

$$\frac{1}{\sec x-\tan x} - \frac{1}{\sec x+\tan x} = ?$$



$[-2\pi, 2\pi, \pi/2]$ by $[-4, 4, 1]$

on Board

Verify the identity. $\dfrac{\cos x-\sin x-1}{\cos x+\sin x+1} = \dfrac{\cos x-1}{\sin x}$

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on Board

Verify the identity.

$$\csc t \cot t = \frac{1+\cot^2 t}{\sec t}$$ on Board

Verify the identity.

$$\frac{\sin\theta+\cos\theta}{\sin\theta}+\frac{\cos\theta+\sin\theta}{\cos\theta}=2+\sec\theta\csc\theta$$

$$\frac{\cos\theta(\sin\theta+\cos\theta)}{\cos\theta\sin\theta}+\frac{(\cos\theta+\sin\theta)\sin\theta}{\cos\theta\sin\theta}$$

$$\frac{\sin\theta\cos\theta+\cos^2\theta}{\sin\theta\cos\theta}+\frac{\sin\theta\cos\theta+\sin^2\theta}{\sin\theta\cos\theta}$$

$$\frac{\sin\theta\cos\theta+\cos^2\theta+\sin\theta\cos\theta+\sin^2\theta}{\sin\theta\cos\theta} \quad =1$$

$$\frac{2\sin\theta\cos\theta+1}{\sin\theta\cos\theta}=\frac{2\cancel{\sin\theta\cos\theta}}{\cancel{\sin\theta\cos\theta}}+\frac{1}{\sin\theta\cos\theta}=2+\frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}=2+\csc\theta\sec\theta$$

Use properties of natural logarithms and fundamental trigonometric identities to show that the pair of expressions is equivalent.

$\ln|1+\cos\theta|-2\ln|\sin\theta|$ and $-\ln|1-\cos\theta|$

$$1-\cos^2\theta$$

$$\sin^2\theta+\cancel{\cos^2\theta-\cos^2\theta}$$

$$\ln|1+\cos\theta|-2\ln\sin\theta=-\ln(1-\cos\theta)$$

$$\ln|1+\cos\theta|-\ln\sin^2\theta \qquad \ln|(1-\cos\theta)^{-1}|=-\ln|1-\cos\theta|$$

$$\ln\left|\frac{1+\cos\theta}{\sin^2\theta}\right|$$

$$\ln\left|\frac{(1+\cos\theta)(1-\cos\theta)}{\sin^2\theta\,(1-\cos\theta)}\right|=\ln\left|\frac{1-\cos^2\theta}{\sin^2\theta(1-\cos\theta)}\right|=\ln\left|\frac{\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}\,(1-\cos\theta)}\right|=\ln\left|\frac{1}{1-\cos\theta}\right|$$

Verify the identity.

$$\frac{\csc(2\theta) - \sin(2\theta)}{\cos(2\theta)} = \cot(2\theta)$$

Verify the identity.

$$\sec^2 x \csc^2 x = \sec^2 x + \csc^2 x$$

On Board

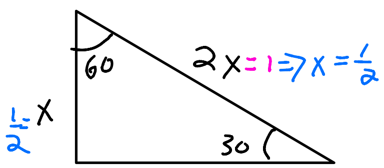
Verify the identity.

$$\frac{\csc\theta \sin\theta}{\cot\theta} = \tan\theta$$

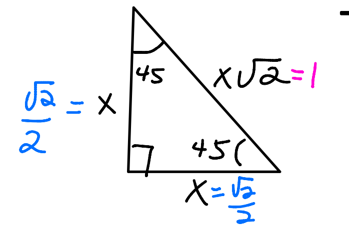
on Board

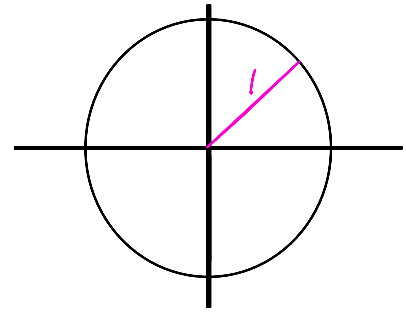
unit circle

radius = 1



$\frac{1}{2} \cdot \frac{\sqrt{3}}{1} = \frac{\sqrt{3}}{2}$





$\frac{x\sqrt{2}}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1\sqrt{2}}{\sqrt{2}\sqrt{2}}$

$x = \frac{\sqrt{2}}{2}$

$8 \cdot \frac{\pi}{12} = \frac{\cancel{4} \cdot 2 \cdot \pi}{\cancel{4} \cdot 3}$

$4 \cdot \frac{\pi}{12} = \frac{4\pi}{3 \cdot \cancel{4}}$

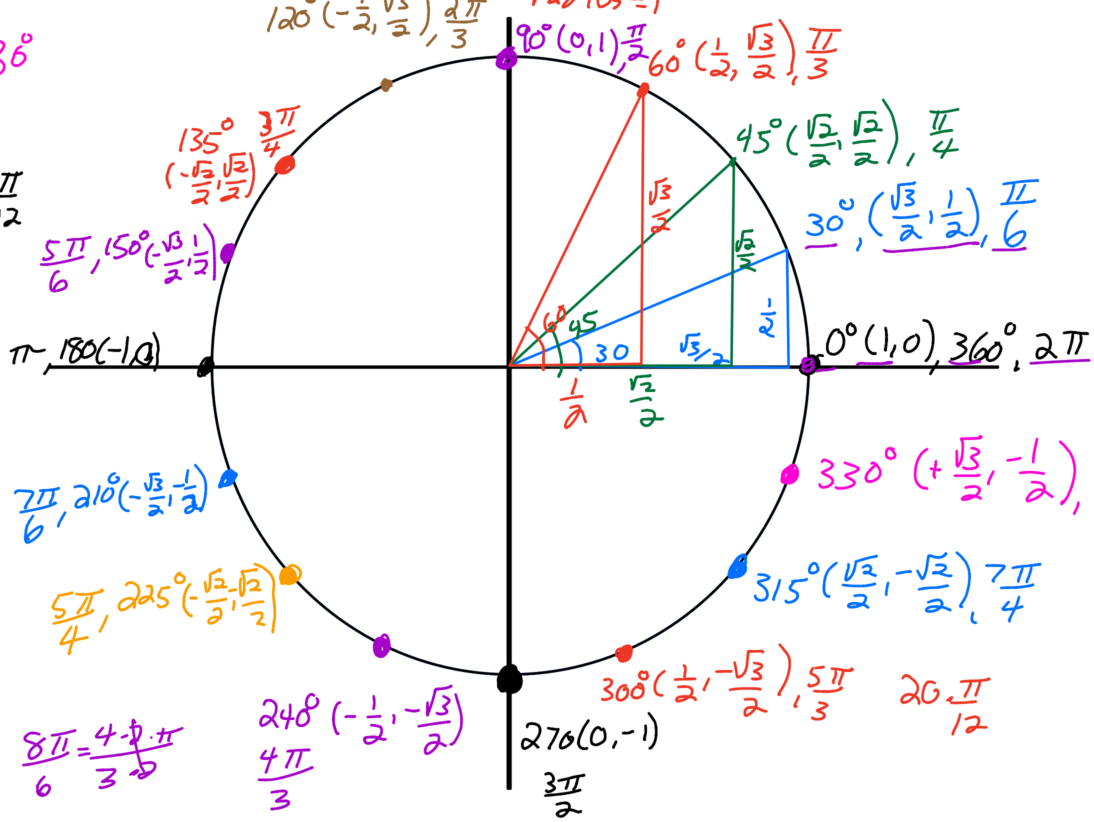
$\frac{\pi}{12} \approx 15°$

$2 \cdot \frac{\pi}{12} = \frac{\pi}{6}$ 30°

$150° \approx 10 \cdot \frac{\pi}{12}$

$150° \approx \frac{5\pi}{6}$

$14 \cdot \frac{\pi}{12}$



$20 \cdot \frac{\pi}{12}$

$\frac{8\pi}{6} = \frac{4 \cdot \cancel{2} \cdot \pi}{3 \cdot \cancel{2}}$

$(\cos\theta, \sin\theta)$

$\cos\frac{2\pi}{3} = -\frac{1}{2}$

$\cos 240° = -\frac{1}{2}$

Use the formula for the cosine of the difference of two angles to find the exact value of the following expression.

cos $(30^\circ - 45^\circ)$

$\cos(a-b) = \cos a \cos b + \sin a \sin B$

$\cos 30 \cos 45 + \sin 30 \sin 45$

$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

$\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$

Addition Formulas
$\sin(A+B) = \sin A \cos B + \cos A \sin B$
$\cos(A+B) = \cos A \cos B - \sin A \sin B$
$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Subtraction Formulas
$\sin(A-B) = \sin A \cos B - \cos A \sin B$
$\cos(A-B) = \cos A \cos B + \sin A \sin B$
$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Verify the identity.

$\sin\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\sin x - \cos x)$

$\sin(a-b) = \sin a \cos b - \sin b \cos a$

$\sin x \cos\frac{\pi}{4} - \sin\frac{\pi}{4} \cdot \cos x$

$\frac{\sqrt{2}}{2} \cdot \sin x - \frac{\sqrt{2}}{2} \cdot \cos x$

$\frac{\sqrt{2}}{2}(\sin x - \cos x)$

Write the expression as the sine, cosine, or tangent of an angle. Then find the exact value of the expression.

$$\frac{\tan\frac{2\pi}{5} + \tan\frac{7\pi}{20}}{1 - \tan\frac{2\pi}{5}\tan\frac{7\pi}{20}} = \frac{\tan a + \tan b}{1 - \tan a \tan b} = \tan(a+b)$$

$$\tan\left(\frac{2\pi \cdot 4}{5 \cdot 4} + \frac{7\pi}{20}\right)$$

$$\tan\left(\frac{8\pi}{20} + \frac{7\pi}{20}\right) =$$

$$\tan\left(\frac{15\pi}{20}\right) = \tan\frac{\cancel{5} \cdot 3 \cdot \pi}{\cancel{5} \cdot 4 \cdot \pi}$$

$$\tan\left(\frac{3\pi}{4}\right) = \frac{\sin\frac{3\pi}{4}}{\cos\frac{3\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \frac{\cancel{\sqrt{2}}}{\cancel{2}} \cdot -\frac{\cancel{2}}{\cancel{\sqrt{2}}} = -1$$

Verify the identity.

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

On Board

$= \frac{3}{5} \cdot \frac{-9}{41} - \frac{4}{5} \cdot \frac{40}{41} = \frac{-27}{205} - \frac{160}{205}$

$\cos\alpha \cos\beta - \sin\alpha \sin\beta = \cos\alpha \cos\beta - \frac{4}{5} \cdot \frac{40}{41}$

Find the exact value of the expressions **cos**(α + β), **sin**(α + β) and **tan**(α + β) under the following conditions:

$\frac{-187}{205}$ $(+,+)$ $(-,+)$

sin(α) = $\frac{4}{5}$, α lies in quadrant I, and **sin**(β) = $\frac{40}{41}$, β lies in quadrant II.

$\sin^2\alpha + \cos^2\alpha = 1$

$\left(\frac{4}{5}\right)^2 + \cos^2\alpha = 1$

$\frac{16}{25} + \cos^2\alpha = 1 - \frac{16}{25}$

$-\frac{16}{25}$

$\cos^2\alpha = \frac{25}{25} - \frac{16}{25} = \frac{9}{25}$

$\sqrt{\cos^2\alpha} = \sqrt{\frac{9}{25}}$

$\cos\alpha = \pm\frac{3}{5} = \frac{3}{5}$

$\sin^2\beta + \cos^2\beta = 1$

$\frac{40}{41} + \cos^2\beta = 1$

$\frac{1600}{1681} + \cos^2\beta = 1 - \frac{1600}{1681}$

$-\frac{1600}{1681}$

$\cos^2\beta = \frac{1681}{1681} - \frac{1600}{1681} = \frac{81}{1681}$

$\sqrt{\cos^2\beta} = \sqrt{\frac{81}{1681}}$

$\cos\beta = \pm\frac{9}{41} = \frac{-9}{41}$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$\frac{4}{5} \cdot \frac{-9}{41} + \frac{40}{41} \cdot \frac{3}{5} = \frac{-36}{205} + \frac{120}{205} = \frac{84}{205} = \frac{7 \cdot 2 \cdot 2 \cdot 3}{41 \cdot 5}$

$\tan(\alpha + \beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\frac{84}{205}}{\frac{-187}{205}} = \frac{84}{\cancel{205}} \cdot \frac{\cancel{205}}{-187} = \frac{84}{-187} = \frac{7 \cdot 2 \cdot 2 \cdot 3}{-11 \cdot 17}$

Find the exact value of each of the following under the given conditions.

$(-,+)$ $(+,+)$

$\tan\alpha = -\frac{4}{3}$, α lies in quadrant II, and $\cos\beta = \frac{5}{6}$, β lies in quadrant I

a. $\sin(\alpha+\beta)$ **b.** $\cos(\alpha+\beta)$ **c.** $\tan(\alpha+\beta)$

$\sin^2\beta + \cos^2\beta = 1$

$\sin^2\beta + \left(\frac{5}{6}\right)^2 = \frac{36}{36}$

$\tan^2\alpha + 1 = \sec^2\alpha$

$\left(-\frac{4}{3}\right)^2 + 1$

$\frac{16}{9} + \frac{9}{9}$

$\frac{25}{9} = \sec^2\alpha = \frac{1}{\cos^2\alpha}$

$\sqrt{\frac{25}{9}} = \sqrt{\frac{1}{\cos^2\alpha}}$

$\pm\frac{5}{3} = \frac{1}{\cos\alpha}$

$-\frac{3}{5} = \cos\alpha$

Find the exact value of the trigonometric expression without the use of a calculator.

$$\sin\left(\sin^{-1}\left(\frac{3}{4}\right)+\cos^{-1}\left(-\frac{4}{9}\right)\right) = \sin(A+B) = \sin A\cos B + \sin B\cos A$$

$$\frac{3}{4}\cdot\frac{-4}{9}+\frac{\sqrt{65}}{9}\cdot\frac{\sqrt{7}}{4} = \frac{-12+\sqrt{455}}{36}$$

$\sin^{-1}\left(\frac{3}{4}\right) = A$

$\sin A = \frac{3}{4}$ $\quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\quad \cos +$

$\cos^{-1}\left(-\frac{4}{9}\right) = B$

$\cos B = -\frac{4}{9}$ $\quad [0, \pi]$ $\quad \sin +$

$\cos A =$

$\sin^2 A + \cos^2 A = 1$

$\left(\frac{3}{4}\right)^2 + \cos^2 A = 1$

$\frac{9}{16} + \cos^2 A = \frac{16}{16} - \frac{9}{16}$

$-\frac{9}{16}$

$\sqrt{\cos^2 A} = \sqrt{\frac{7}{16}}$

$\cos A = +\frac{\sqrt{7}}{4}$

$\sin^{-1} a = x$	$x \Rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\csc^{-1} a = x$	$x \Rightarrow \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$\tan^{-1} a = x$	$x \Rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cos^{-1} a = x$	$x \Rightarrow [0, \pi]$
$\sec^{-1} a = x$	$x \Rightarrow \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$\cot^{-1} a = x$	$x \Rightarrow (0, \pi)$

$\cos^2 B + \sin^2 B = 1$

$\left(-\frac{4}{9}\right)^2 + \sin^2 B = 1$

$\frac{16}{81} + \sin^2 B = \frac{81}{81} - \frac{16}{81}$

$-\frac{16}{81}$

$\sqrt{\sin^2 B} = \sqrt{\frac{65}{81}}$

$\sin B = \frac{\sqrt{65}}{9}$